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Numerical assessment of thermo-acoustic instabilities in gas turbines

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SUMMARY

A new methodology to assess the effect of the flame/acoustics coupling on the stability of the modes without combustion is presented. An asymptotic method is used to account for the acoustic flame transfer function. The efficiency and accuracy of the approach is demonstrated on an academic case similar to a Rijke tube configuration. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: acoustics; combustion instabilities; eigenmodes

1. INTRODUCTION

It has been known for a long time that the coupling between acoustic waves and flames in industrial systems can lead to high amplitude instabilities [1–3]. In addition to inducing oscillations of all physical quantities (pressure, velocities, temperature, etc.), these instabilities can increase the amplitude of the flame motion and, in extreme cases, destroy part of the burner due to large heat transfer in the premixing tube. Since the equivalence ratio oscillates when instabilities are present, there is a general trend for combustors to be more unstable when operating in the lean regime (more air injected than necessary to burn the amount of fuel injected). Besides, due to new international constraints, pollutant emissions must be reduced and many gas turbine manufacturers' strategies consist in operating their systems under leaner and leaner conditions. Consequently, there is a need to better understand combustion instabilities and to predict them at the design level. The objective of this paper is to present a methodology to predict unstable/stable thermo-acoustic modes of a combustor. Since no assumptions about the geometry are required, this method can be applied to realistic configurations [4]. The equations of linear acoustics are first written in the case of three-dimensional reactive flows, and the problem is closed by using flame transfer functions (which can be

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evaluated with large Eddy simulation calculations on realistic configurations [5]). An asymptotic expansion method is then developed in order to recover a classical eigenvalues problem from these equations. Finally, the methodology is tested by computing an academic example whose theoretical solution is known.

2. METHODOLOGY

2.1. Governing equations

A suitable description of the thermo-acoustic instabilities can be derived by making use of the perfect gas law and classical equations of fluid mechanics, i.e. equations of mass, momentum and energy conservation. Besides, assumptions of constant mean pressure and low Mach number appear reasonable from gas turbine observations. Moreover, since eigenmodes exhibited in practical systems lie in the low/medium frequency domain, viscosity as well as thermodiffusivity may be neglected. Under these assumptions, a wave equation for small pressure perturbations may be derived [6] and reads

$$\nabla(\bar{c}^2\nabla p') - \frac{\partial^2 p'}{\partial t^2} = -(\gamma - 1)\frac{\partial \dot{q}'}{\partial t} \tag{1}$$

where primed and overbarred variables stand for the thermo-acoustic perturbation and mean variables, respectively, whereas p, c and $\dot{q}$ stand for pressure, sound speed and rate of heat release. Note that the specific heat ratio γ has been assumed constant for deriving this equation but the flow field fluctuations are not supposed isentropic. Equation (1) is thus relevant to any large scale of small amplitude pressure fluctuations. Solving this equation requires a model for the rate of heat release fluctuations $\dot{q}'$ in order to close the problem. As suggested by the seminal studies of Crocco [7, 8], the flame is modelled as a purely acoustic element, neglecting the effects of local turbulence, chemistry or heat losses. The simplest model reads

$$\frac{\dot{q}'(\mathbf{x}, t)}{\bar{\dot{q}}(\mathbf{x})} = n_l(\mathbf{x})\frac{\mathbf{u}'(\mathbf{x}_{\text{ref}}, t - \tau(\mathbf{x}))\cdot\mathbf{n}_{\text{ref}}}{\bar{\mathbf{u}}(\mathbf{x}_{\text{ref}})\cdot\mathbf{n}_{\text{ref}}} \tag{2}$$

where $n_l(\mathbf{x})$ is a local interaction index and $\tau(\mathbf{x})$ stands for a time lag between the local unsteady heat release $\dot{q}'(\mathbf{x}, t)$ and the acoustic velocity $\mathbf{u}'$ at a reference position $\mathbf{x}_{\text{ref}}$ and direction $\mathbf{n}_{\text{ref}}$. The other variables introduced are the local mean rate of heat release $\bar{\dot{q}}(\mathbf{x})$ and the mean speed at the reference point $\bar{\mathbf{u}}(\mathbf{x}_{\text{ref}})$. This formulation generalizes the n–τ model [7, 8] used in the framework of one-dimensional configurations with infinitely thin flames [6] to the case of three-dimensional flows with distributed combustion. Assuming harmonic fluctuations of small amplitudes, $p' = \Re(\hat{p}(\mathbf{x})e^{-i\omega t})$, with $i^2 = -1$, Equations (1) and (2) can be combined with the linearized momentum equation $i\omega\bar{\rho}\,\hat{\mathbf{u}} = \nabla\hat{p}$ to give

$$\nabla(\bar{c}^2\nabla\hat{p}) + \omega^2\hat{p} = \frac{(\gamma - 1)\bar{\dot{q}}(\mathbf{x})}{\bar{\rho}(\mathbf{x}_{\text{ref}})\,\bar{\mathbf{u}}(\mathbf{x}_{\text{ref}})\,\mathbf{n}_{\text{ref}}}\,n_l(\mathbf{x})e^{i\Re(\omega)\tau(\mathbf{x})}\nabla\hat{p}\cdot\mathbf{n}_{\text{ref}}(\mathbf{x}_{\text{ref}}) \tag{3}$$

This latter equation together with proper boundary conditions constitutes the eigenvalue problem satisfied by the harmonic fluctuations $\hat{p}$ in the flow domain Ω bounded by the surface

$\partial\Omega = \partial\Omega_D \cup \Omega_{VN} \cup \Omega_Z$. Three types of boundary conditions have been considered:

- A Dirichlet condition $\hat{p} = 0$ on $\partial\Omega_D$.
- A Neumann condition $\hat{\mathbf{u}} \cdot \mathbf{n} = 0$, $\mathbf{n}$ the outward normal unit vector on $\partial\Omega_N$.
- An admittance type condition on $\partial\Omega_Z$ which implies

$$\frac{1}{Z} = \frac{-\mathrm{i}\bar{c}\nabla\hat{p}\cdot\mathbf{n}}{\omega\hat{p}} \tag{4}$$

where Z is the reduced impedance $Z = \hat{p}/(\bar{\rho}\bar{c}\,\hat{\mathbf{u}}\cdot\mathbf{n})$.

2.2. Numerical approach

At first, the problem without source term is considered. This corresponds to the case of an acoustically passive flame with zero unsteady heat release $\dot{q}'$. Note, however, that the mean heat release $\bar{\dot{q}}$ is not necessarily zero. Consistently, the mean temperature and speed of sound may still be functions of space. Using the classical Galerkin finite element method to discretize the problem and assuming $1/Z = \alpha_1/\omega + \alpha_2 + \alpha_3\omega$, α_1, α_2, α_3, complex constants, one ends up with the finite dimension problem

$$[A][P] + \omega[B][P] + \omega^2[P] = 0 \tag{5}$$

where the matrix $[A]$ of size m represents the $\nabla(\bar{c}^2\nabla)$ operator and $[B]$ represents the boundary terms. $[P]$ is the column vector whose components are the values of $\hat{p}$ at the m nodes of the finite element mesh. The resulting problem is not linear anymore (with respect to ω^2) but this difficulty can be overcome by using a suitable variable transformation [9]. A classical eigenvalue problem of size $2\times m$ can then be recovered and solved by an Arnoldi method [10].

2.3. Accounting for the unsteady combustion

The eigenvalue problem associated with Equation (3) cannot be solved directly by classical methods developed for linear algebra. In the present approach, the flame is considered as an element which slightly modifies the eigenmode without combustion. Specifically, a global energy form of Equation (3) is first derived by multiplying this equation by $\hat{p}$ and integrating over Ω

$$\int_\Omega \hat{p}[\nabla(\bar{c}^2\nabla\hat{p}) + \omega^2\hat{p}]\,\mathrm{d}V = \int_\Omega \frac{(\gamma-1)\bar{\dot{q}}(\mathbf{x})}{\bar{\rho}(\mathbf{x}_{\mathrm{ref}})\,\bar{\mathbf{u}}(\mathbf{x}_{\mathrm{ref}})\cdot\mathbf{n}_{\mathrm{ref}}}\, n_l(\mathbf{x})\mathrm{e}^{\mathrm{i}\Re(\omega)\tau(\mathbf{x})}\hat{p}\nabla\hat{p}\cdot\mathbf{n}_{\mathrm{ref}}(\mathbf{x}_{\mathrm{ref}})\,\mathrm{d}V \tag{6}$$

We then define the expansion parameter $\varepsilon = \frac{1}{V_\Omega}\int_\Omega n_l(\mathbf{x})\,\mathrm{d}V$ and seek for the eigenmodes $(\omega, \hat{p})$ of Equation (3) as a first-order expansion around the modes without combustion $(\omega_0, \hat{p}_0)$

$$\omega = \omega_0 + \varepsilon\omega_1 + o(\varepsilon^2) \tag{7}$$

$$\hat{p} = \hat{p}_0 + \varepsilon\hat{p}_1 + o(\varepsilon^2) \tag{8}$$

Introducing these relations in Equation (6) and keeping only first-order terms give the following equation:

$$\int_{\Omega} \hat{p}_0[\nabla(\bar{c}^2\nabla\varepsilon\hat{p}_1) + \omega_0^2\varepsilon\hat{p}_1]\,\mathrm{d}V = -2\varepsilon\int_{\Omega} \hat{p}_0^2\omega_0\omega_1\,\mathrm{d}V + \int_{\Omega} \frac{(\gamma-1)\bar{\bar{q}}(\mathbf{x})(\nabla\hat{p}_0\cdot\mathbf{n}_{\mathrm{ref}})(\mathbf{x}_{\mathrm{ref}})}{\bar{\rho}(\mathbf{x}_{\mathrm{ref}})\,\bar{\mathbf{u}}(\mathbf{x}_{\mathrm{ref}})\cdot\mathbf{n}_{\mathrm{ref}}} n_1(\mathbf{x})\mathrm{e}^{\mathrm{i}\Re(\omega_0)\tau(\mathbf{x})}\hat{p}_0\,\mathrm{d}V \tag{9}$$

The LHS term can be simplified by using a reduction order method [11] in which $\hat{p}_1 = \hat{p}_0 F_1$, F_1 being a spatial derivable function. Owing to this relation, the LHS term of Equation (9) becomes

$$\int_{\Omega} \hat{p}_0[\nabla\cdot(\bar{c}^2\nabla\varepsilon\hat{p}_1) + \omega_0^2\varepsilon\hat{p}_1]\,\mathrm{d}V = \varepsilon\int_{\partial\Omega} \bar{c}^2\,\hat{p}_0^2\nabla\mathbf{F_1}\,\mathrm{d}\mathbf{S} \tag{10}$$

which is obviously null on $\partial\Omega_D$ since $\hat{p}_0 = 0$. Moreover, the eigenmodes with flame $(\omega, \hat{p})$ and without flame $(\omega_0, \hat{p}_0)$ verify the same boundary conditions, one can show that $\nabla F_1\,\mathrm{d}\mathbf{S} = 0$ on $\partial\Omega_N$ and that the following relation is valid at the first-order in ε on $\partial\Omega_Z$

$$\nabla F_1\,\mathrm{d}\mathbf{S} = \frac{\mathrm{i}\omega_1}{\bar{c}Z(\omega_0)}\left(1 - \frac{1}{Z(\omega_0)}\frac{\partial Z}{\partial\omega}(\omega_0)\right) \tag{11}$$

Consequently, by introducing this relation in the RHS term of Equation (10), an expression for the perturbation $\varepsilon\omega_1$ can be obtained

$$\varepsilon\omega_1 = \frac{\int_{\Omega}\bar{\bar{q}}(\mathbf{x})\hat{p}_0 n_1(\mathbf{x})\mathrm{e}^{\mathrm{i}\Re(\omega_0)\tau(\mathbf{x})}(\gamma-1)(\nabla\hat{p}_0\cdot\mathbf{n}_{\mathrm{ref}})(\mathbf{x}_{\mathrm{ref}})\,\mathrm{d}V}{\bar{\rho}(\mathbf{x}_{\mathrm{ref}})\,\bar{\mathbf{u}}(\mathbf{x}_{\mathrm{ref}})\cdot\mathbf{n}_{\mathrm{ref}}\left[2\omega_0\int_{\Omega}\hat{p}_0^2\,\mathrm{d}V + \int_{\partial\Omega_Z}\frac{\mathrm{i}\bar{c}\hat{p}_0^2}{Z(\omega_0)}(1 - 1/Z(\omega_0)\partial Z/\partial\omega(\omega_0))\,\mathrm{d}S\right]} \tag{12}$$

In the case where the denominator of Equation (12) is not null, this equation provides a simple way to check whether an eigenmode without combustion $(\omega_0, \hat{p}_0)$ is made stable $(\Im(\omega_0 + \varepsilon\omega_1) < 0)$ or unstable $(\Im(\omega_0 + \varepsilon\omega_1) > 0)$ by the coupling with the unsteady flame.

3. ACADEMIC EXAMPLE OF APPLICATION

3.1. Description of the configuration and theoretical solution

The aforementioned method is tested on the configuration illustrated by Figure 1. It deals with a two-dimensional tube with a closed inlet, an open outlet and a mean temperature jump induced by a flame located at its middle. Since the flame thickness is much smaller than the typical wavelength, the flame is considered as infinitely thin. Following the methodology of Poinsot and Veynante [6], suitable jump relations across the flame provide a characteristic relation matched by the pulsation ω of the longitudinal modes,

$$\cos\left(\frac{kL}{4}\right), \left[n\mathrm{e}^{\mathrm{i}\omega\tau}\sin^2\left(\frac{kL}{4}\right) - 3\cos^2\left(\frac{kL}{4}\right) + 2\right] = 0 \tag{13}$$

where $k = \omega/c_1$ stands for the wave number in the fresh gases. With the formalism chosen in the second section, an eigenmode is unstable whenever $\Im(\omega)$ is positive. Besides, the classical

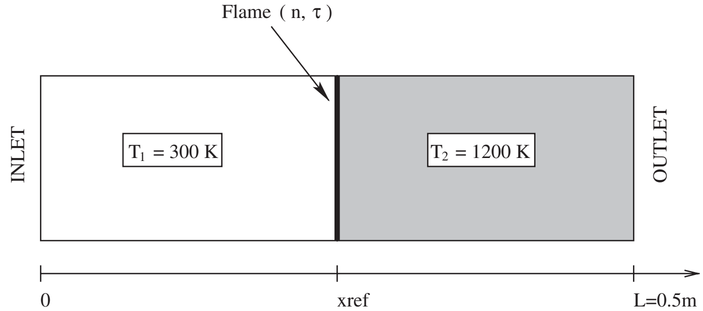

Figure 1. Configuration retained for first-order expansion method validation.

one-dimensional n–τ model [6] used in Equation (13) can be related to the model in Equation (2) and

$$\varepsilon = \frac{\bar{c}^2(\mathbf{x}_{\mathrm{ref}})}{(\gamma - 1)c_p(\bar{T}_2 - \bar{T}_1)}\, n \tag{14}$$

where c_p is the massic heat capacity at constant pressure. In this simple example, c_p and γ are considered spatially constant and are equal to $1004.5\,\mathrm{J\,K^{-1}\,kg^{-1}}$ and 1.4, respectively. In addition, following Crocco [7, 8], the heat release fluctuations are coupled to the velocity fluctuations in the fresh gas. In this academic example, it means that the theoretical reference position required for defining the flame transfer function is chosen immediately upstream the flame, i.e. at the abscissa $x_{\mathrm{ref}} = 0.25\,\mathrm{m}$.

3.2. Application of the asymptotic expansion method

Although the configuration described in Section 3.1 is one-dimensional, all the calculations have been performed on unstructured two dimensional meshes with triangular cells. Two kinds of mesh have been used: a first one with 561 nodes and a second one, highly refined in the flame vicinity, with 5231 nodes. Two main issues have been addressed:

- *Optimal position of the reference point*: Because of the large temperature variation near x_{ref}, the computation of the acoustic pressure gradient at the reference point (see Equation (12)) is not reliable in the vicinity of the flame. To overcome this difficulty, the reference location has been taken as the closest grid point to the flame in the fresh gas, *viz*. $x_{\mathrm{ref}} = 0.24\,\mathrm{m}$ for the 561 nodes mesh instead of $x_{\mathrm{ref}} = 0.25\,\mathrm{m}$ for the theoretical model. The refinement in the second mesh allows a reference position closer to the theoretical value: $x_{\mathrm{ref}} = 0.249\,\mathrm{m}$. Table I shows that the results obtained with the two meshes are in close agreement with the theory: as long as the gap between the theoretical and the numerical reference location is small in comparison with the eigenmode wavelength, accurate computation can be performed.
- *Validity domain with respect to expansion parameter ε value*: Three values of the expansion parameter are considered, $\varepsilon = 0.003$ ($n = 0.01$), $\varepsilon = 0.33$ ($n = 1.0$) and $\varepsilon = 1.6$ ($n = 5.0$) the time delay in any case being $\tau = 10^{-4}\,\mathrm{s}$. Eigen frequencies obtained in each case are compared with theoretical solutions of Equation (13). The results are

Table I. Effect of reference position value and the grid resolution for the first eigen frequency; $\varepsilon = 0.003$, $\tau = 10^{-4}$ s. Cross 'X' indicates unfeasible calculation.

	$x_{\text{ref}} = 0.24\,\text{m}$	$x_{\text{ref}} = 0.249\,\text{m}$	$x_{\text{ref}} = 0.25\,\text{m}$
Coarse mesh (561 nodes)	$270.4 - 0.087i$	X	X
Refined mesh (5231 nodes)	$271.3 - 0.093i$	$271.4 - 0.088i$	X
Theoretical	$271.5 - 0.098i$	$271.6 - 0.088i$	$271.6 - 0.088i$

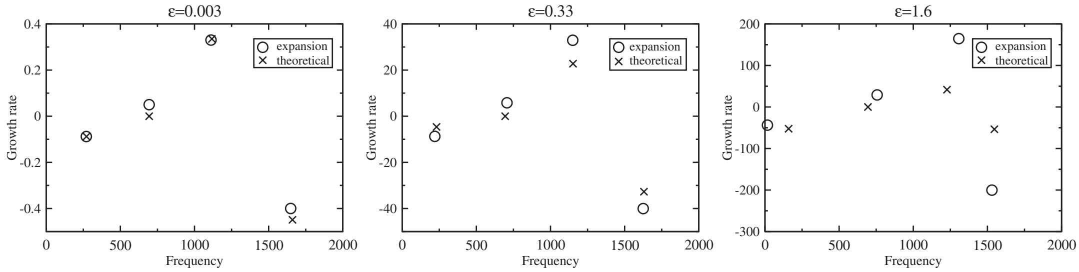

Figure 2. Representation in the complex plane of the theoretical and computed eigen frequencies.

available in Figure 2 and displayed in the complex plane. As expected, the computed eigen frequencies match the theoretical results for low values of ε. Discrepancies appear for ε values close to one but the error on the real part of the eigen frequencies is bounded to 15% except for the first mode with $\varepsilon = 1.6$. Concerning the imaginary part of the eigen frequencies, the error is more important but the stability of the mode, i.e. the sign of the imaginary part, is always correctly predicted. Moreover, the third mode is always found the most unstable. From these results, first-order asymptotic expansion seems to increase the shift induced by the flame in eigen frequency values but the trend is correctly predicted even for expansion parameter values beyond its theoretical application range ($\varepsilon \ll 1$).

4. CONCLUSION

A methodology to evaluate the stability of the thermo-acoustic eigenmodes with an active acoustic flame has been proposed. Because of the particular source term induced by the flame, a special treatment is required. An asymptotic expansion method used together with a flame transfer function model allows the assessment of how the unsteady combustion modifies the stability of the eigenmodes of the system. This strategy is tested on an academic case. The method accurately assesses the eigen frequencies for low to medium values of the expansion parameter ε and seems to predict correctly the trend when $\varepsilon > 1$. Further developments will include the application to realistic gas-turbine configurations.

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